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INVESTIGATION OF THERMODYNAMIC GEOMETRY AND CHRISTOFFEL COEFFICIENTS OF MODIFIED CHAPLYGIN GAS USING WEINHOLD METRIC

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ABSTRACT

In this article, the dynamic components and temperature values of modified Chaplygin gas, which has the equation of state $P = A\rho - \frac{B}{\rho^{\alpha}}$ were obtained as a function of entropy and volume for the universe filled with modified Chaplygin gas, first at $\alpha = 0$ and then at $\alpha = 1$. All symbols used in the derivation are defined explicitly, and the entropy-dependent integration functions are fixed through the thermodynamic relation T=(∂U/∂S)V. The original contribution of this work is the combined derivation of the Weinhold curvature and the corresponding Christoffel coefficients for modified Chaplygin gas within the same thermodynamic phase-space framework. The signs of the thermodynamic curvature (R=0, R>0, R<0) were interpreted only as thermodynamic-geometric indicators of the dominant interaction regime and were compared with the related literature. A numerical illustration is also included to show how the curvature sign changes with model parameters. Subsequently, all components of the Weinhold metric and the thermodynamic curvature for modified Chaplygin gas were obtained for both $\alpha = 0$ and $\alpha = 1$. Finally, the Christoffel coefficients were obtained on the Weinhold thermodynamic phase space for the same conditions.

1. INTRODUCTION

Observations made by Edwin Hubble in the early 20th century revealed that the universe is expanding and galaxies are moving away from each other [1].

Recent astronomical observations, such as Type Ia supernovae (SNe Ia), large-scale redshift surveys (LSS), cosmic microwave background radiation (CMB), and the Wilkinson Microwave Anisotropy Probe (WMAP), have shown that the universe is not only expanding but that this expansion is accelerating [2-15].

The accelerating expansion of the universe can be explained by dark energy, defined as an exotic fluid with positive energy density and negative pressure [16-18].

Cosmologists have proposed many dark energy candidates. The first and simplest of these is the cosmological constant Λ [19-21].

Later, various candidates for dark energy were proposed, such as Cosmological constant, quintessence, k-essence, tachyon, phantom, Chaplygin gas, quintom, holographic dark energy, and extra dimensions [22-30].

The dark energy and dark matter portion of the universe is still not fully explained. Considering dark energy and dark matter as a single component is only possible with Chaplygin gas. In recent years, Santos et al. have investigated thermodynamic stability in the modified Chaplygin gas model [31,32].

Pure Chaplygin gas or modified Chaplygin gas is a fluid that is thought to behave like a pressureless fluid in the early universe and like the cosmological constant in the later stages [33].

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The novelty of the present work is not the introduction of a new dark-energy model, but the thermodynamic-geometric treatment of modified Chaplygin gas using the Weinhold metric for both α=0 and α=1, together with the explicit calculation of the corresponding Christoffel coefficients. This provides a compact way to connect thermodynamic curvature, stability-related quantities, and phase-space geodesic structure in the same framework.

Recently, the development of quantum sensing systems in the field of quantum technologies has made it possible to determine fundamental thermodynamic parameters such as entropy and temperature. If the geometric effects of dark energy on spacetime can be obtained based on statistical and mechanical quantities such as entropy, temperature, and volume, then data on the physical properties of dark energy can also be obtained through quantum detectors [34-37].

This could also pave the way for quantum field theory to be evaluated in terms of general relativity and statistical physics.

Weinhold has made significant contributions to thermodynamics. The metric he proposed for thermodynamics is as follows:

$$\eta_U = \sum_{ij} \frac{\partial U}{\partial x_i \partial x_j} dx_i dx_j \tag{1}$$

Here, U is the energy of the thermodynamic system, $X_1 = S$ is the entropy, and $X_2 = V$ is the volüme [38].

When the entropy S is taken instead of the energy U of the thermodynamic system in equation (1), the Ruppeiner metric

$$\eta_U = -\frac{1}{T}\eta_S \tag{2}$$

It is in the form of. Here, η_U Weinhold metric, η_S Ruppeiner metric and T defines the temperature [39,40].

In the first part of this study, the dynamic components of modified Chaplygin gas ($P = A\rho - \frac{B}{\rho^\alpha}$) were obtained as a function of temperature, pressure, and volume for the cases $\alpha = 0$ and $\alpha = 1$. The entropy-dependent integration functions appearing in U(S,V) were stated explicitly because they determine the temperature, Weinhold metric components, and curvature. In the second part, the components of the Weinhold metric and the thermodynamic curvature of the modified Chaplygin gas were calculated for the cases $\alpha = 0$ and $\alpha = 1$. Finally, all Christoffell coefficients of the modified Chaplygin gas were obtained for the cases $\alpha = 0$ and $\alpha = 1$. And a numerical illustration is given to support the analytical discussion and to clarify the parameter dependence of the model.

2. THERMODYNAMIC GEOMETRY AND SPACE-TIME CURVATURE OF MODIFIED CHAPLYGIN GAS

Dark energy, an exotic structure thought to constitute 70% of the universe according to many observational and theoretical studies, still remains a mystery [41]. The equation of state of modified Chaplygin gas V, one of the dark energy candidates, is as follows [42-44].

$$P = A\rho - \frac{B}{\rho^\alpha} \tag{3}$$

Here, P, is the pressure of the modified Chaplygin gas, and ρ is the density of the modified Chaplygin gas. $A > 0$ and $B > 0$ and $0 \leq \alpha \leq 1$ dır.

$\alpha = 0$ in case of;

$$p = A\rho - B \tag{4}$$

is obtained. Also ρ

$$\rho = \frac{U}{V} \tag{5}$$

It can be expressed as follows. Here U is energy and V is volüme [45,46].

To express the thermodynamic curve of modified Chaplygin gas, its energy U, pressure P, and temperature T are denoted as a function of entropy S and volume V. From the laws of thermodynamics

$$\left(\frac{\partial U}{\partial V}\right)_S = -p \tag{6}$$

$$\left(\frac{\partial U}{\partial S}\right)_V = T \tag{7}$$

is obtained as follows [32].

For modified Chaplygin gas from equations (4), (5) and (6)

$$\left(\frac{\partial U}{\partial V}\right)_S = -\frac{AU}{V} + B \tag{8}$$

is obtained from the solution of equation (8).

$$U = \frac{BV}{A+1} + V^{-A}c \tag{9}$$

Here c is a constant; it is an entropy-dependent integration function $c(S)$ This choice is required because Eq. (8) is integrated with respect to V at fixed S, and the remaining function of S is then fixed by the thermodynamic relation $T = \left(\frac{\partial U}{\partial S}\right)V$ and by the entropy normalization used in Ref. [32].

Substituting equation (9) into equation (7) provides the temperature for modified Chaplygin gas.

$$T = V^{-A}\left(\frac{dc}{dS}\right) \tag{10}$$

In equation (10), the function $c(S)$must be determined explicitly in terms of entropy S. Following the normalization convention used for modified Chaplygin gas in Ref. [32], the entropy is related to c by

$$S = \frac{2}{\lambda}c^{\frac{1}{2}} \tag{11}$$

is in the form [32]. Here λ is a constant. Temperature from equations (10) and (11)

$$T = \frac{S\lambda^2}{2V^A} \tag{12}$$

This is how it is obtained. Thus, the temperature T of the modified Chaplygin gas is obtained as a function of entropy and volume. Entropy

$$S \equiv \frac{(\rho + p)V}{T} \tag{13}$$

entropy is in the form of [30]. (4), (12), (13) equations

$$S = \frac{2\left(\frac{B+p}{A} + p\right)V^{1+A}}{S\lambda^2} \tag{14}$$

The expression of pressure p as a function of entropy and volume is obtained from equation (14).

$$p = \frac{V^{-A}S^2\lambda^2 - 2V\rho}{2V} \tag{15}$$

The Weinhold metric is required for the thermodynamic curvature of the modified Chaplygin gas. From equation (1), the Weinhold metric is as follows [47].

$$\eta_{ij} = \begin{pmatrix} \frac{T}{c_V} & -\frac{T\alpha}{c_V k} \\ -\frac{T\alpha}{c_V k} & \frac{c_p}{c_V V k} \end{pmatrix} \tag{16}$$

Here

$$c_V = T\left(\frac{\partial S}{\partial T}\right)_V \tag{17}$$

It is the heat capacity at constant volume.

$$c_p = T\left(\frac{\partial S}{\partial T}\right)_\rho \tag{18}$$

It is the heat capacity at constant pressure.

$$\alpha = \frac{1}{V}\left(\frac{dV}{dT}\right)_p \tag{19}$$

It is the volumetric expansion coefficient.

$$k = -\frac{1}{V}\left(\frac{dV}{dp}\right)_T \tag{20}$$

Isothermal compressibility. These equations are related to the following equation.

$$c_{p-}c_V = VT\frac{\alpha^2}{k} \tag{21}$$

If equation (19) is substituted into equation (21)

$$\frac{\alpha}{k} = \frac{\left(c_p - c_V\right)}{T\left(\frac{dV}{dT}\right)_p} \tag{22}$$

The components of the Weinhold metric for modified Chaplygin gas are obtained by using equations (12) and (15) in equations (16)-(22).

$$\eta_{11} = \frac{T}{c_V} = \frac{S\lambda^2}{2V^A c_V} \tag{23}$$

$$\eta_{12} = \eta_{21} = \frac{S\lambda^2 A V^{-(1+A)}\left(c_p - c_V\right)}{2c_V} \tag{24}$$

$$\eta_{22} = -\frac{S\lambda^2 A^2 V^{-(2+A)} c_p\left(c_p - c_V\right)}{2c_V} \tag{25}$$

The thermodynamic curvature of the modified Chaplygin gas is calculated using the appropriate thermodynamic metric formalism. Thermodynamic curvature

$$R = -\frac{1}{4det(\eta)^2} det\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \tag{26}$$

is in the form of [48]. Using equations (23)-(25), the components of the determinant for the modified Chaplygin gas are obtained as follows.

$$\eta_{11,1} = \frac{\partial \eta_{11}}{\partial S} = \frac{\lambda^2}{2c_V V^A} \tag{27}$$

$$\eta_{11,2} = \frac{\partial \eta_{11}}{\partial V} = -\frac{S\lambda^2 A}{2c_V V^{1+A}} \tag{28}$$

$$\eta_{12,1} = \frac{\partial \eta_{12}}{\partial S} = \frac{\lambda^2 A\left(c_p - c_V\right)}{2c_V V^{1+A}} \tag{29}$$

$$\eta_{12,2} = \frac{\partial \eta_{12}}{\partial V} = -\frac{S\lambda^2 A(A+1)\left(c_p - c_V\right)}{2c_V V^{A+2}} \tag{30}$$

$$\eta_{22,1} = \frac{\partial \eta_{22}}{\partial S} = -\frac{\lambda^2 A^2 c_p\left(c_p - c_V\right)}{2c_V V^{A+2}} \tag{31}$$

$$\eta_{22,2} = \frac{\partial \eta_{22}}{\partial V} = \frac{S\lambda^2 A^2 (A+2) c_p\left(c_p - c_V\right)}{2c_V V^{A+3}} \tag{32}$$

The value of the determinant using equations (27)-(32)

$$det(\eta) = -\frac{S^2\lambda^4 A^2\left(c_p - c_V\right)\left(2c_p - c_V\right)}{4{c_V}^2 V^{2+2A}} \tag{33}$$

$$det = -\frac{1}{8{c_V}^3 V^4}\left[S\lambda^4 A^2\left(c_p - c_V\right) D_1\right] \tag{34}$$

Here, $D_1 = \left(S\lambda^2\left(c_p - c_V\right)(A+1)V^{1-3A} - 2(S+1)c_V V^{2(1-A)} + ASc_p\lambda^2 V^{-3A}\left(c_p - c_V\right)\right)$ it is in the form of.

When we substitute equations (27)-(34) into equation (26), the thermodynamic curvature of the modified Chaplygin gas is obtained as a function of temperature and entropy as follows.

$$R = \frac{c_V\left(-2c_V(1+S)V^{2(1+A)} + S\lambda^2\left(c_p - c_V\right)\left[(1+A)V^{1+A} + Ac_p V^A\right]\right)}{8S^3\lambda^4 A^2\left(c_p - c_V\right)\left(c_p - \frac{c_V}{2}\right)^2} \tag{35}$$

Equations (34) and (35) show that the scalar curvature of modified Chaplygin gas is nonzero in general. In thermodynamic geometry, the sign and magnitude of this curvature are commonly associated with the effective interaction structure of the underlying thermodynamic system. A vanishing curvature corresponds to an ideal-gas-like limit, positive curvature is usually linked with repulsive effective interactions, and negative curvature is usually linked with attractive effective interactions.

$\alpha = 1$ in case of; The equation of state for modified Chaplygin gas is as follows.

$$p = A.\rho - \frac{B}{\rho} \tag{36}$$

Here, p is the pressure of the modified Chaplygin gas, and ρ is its density. $A > 0$ and $B > 0$.

Equations (5), (6) and (36) are obtained for modified Chaplygin gas.

$$\left(\frac{\partial U}{\partial V}\right)_S = -\frac{AU}{V} + \frac{BV}{U} \tag{37}$$

It is obtained from the solution of equation (37).

$$U = \frac{\sqrt{(A+1)(V^{-2A}c_1 A + BV^2 + V^{-2A}c_1}}{A+1} \tag{38}$$

Here, c_1 is a constant and a function of the entropy S. Here, c_1 is again an entropy-dependent integration function, rather than an arbitrary constant. As in the α=0 case, its S-dependence is fixed after the V-integ $c_1(S)$, ration by imposing $T = \left(\frac{\partial U}{\partial S}\right)V$ and the same entropy normalization.

Substituting equation (38) into equation (7) gives the temperature for modified Chaplygin gas.

$$T = \frac{V^{-2A}\sqrt{A+1}\frac{dc_1}{dS}}{2\sqrt{c_1(A+1)V^{-2A} + BV^2}} \tag{39}$$

After specifying the entropy-dependent integration function through this thermodynamic normalization, substituting equation (38) into equation (7) gives the temperature of modified Chaplygin gas as

$$T = \frac{V^{-2A} S\lambda^2 \sqrt{A+1}}{2\sqrt{S^2\lambda^2(A+1)V^{-2A} + 4BV^2}} \tag{40}$$

Thus, the temperature of the modified Chaplygin gas is obtained as a function of entropy and volume. Equations (13), (36) and (40) are in the form of entropy.

$$S = \frac{p(1+2A)\sqrt{4AB+p^2}\sqrt{S^2\lambda^2(1+A)V^{-2A} + 4BV^2}}{S\lambda^2 A\sqrt{1+A}V^{-2A}} \tag{41}$$

From equation (41), p is obtained as pressure.

$$\begin{aligned} p = {} & \frac{V^{-A}\sqrt{16BS^2\lambda^2(1+A)V^{2A+2} + \lambda^4 S^4 + 64B^2V^{4+4A}}}{4\sqrt{4BV^{2A+2} + S^2\lambda^2(1+A)}\sqrt{1+A}\,V} \\ & - \frac{-2V^{-2A}\lambda^2 S^2\left(A+\frac{1}{2}\right)}{4\sqrt{S^2\lambda^2(1+A)V^{-2A} + 4BV^2}\sqrt{1+A}\,V} \\ & - \frac{V^{-A}\sqrt{16BS^2\lambda^2(1+A)V^{2A+2} + \lambda^4 S^4 + 64B^2V^{4+4A}}}{4\sqrt{4BV^{2A+2} + S^2\lambda^2(1+A)}\sqrt{1+A}V} \\ & + \frac{V^{-2A}\lambda^2 S^2\left(A+\frac{1}{2}\right)}{2V\sqrt{S^2\lambda^2(1+A)V^{-2A} + 4BV^2\sqrt{1+A}}} \end{aligned} \tag{42}$$

Using equations (40) and (42) in equations (16)-(22), the components of the Weinhold metric for modified Chaplygin gas are obtained as follows.

$$\eta_{11} = \frac{V^{-2A} S\lambda^2\sqrt{A+1}}{2\sqrt{S^2\lambda^2(A+1)V^{-2A} + 4BV^2}c_V} \tag{43}$$

$$\eta_{12} = \eta_{21} = \frac{\left(c_p - c_V\right)\sqrt{A+1}\lambda^2 S\left(S^2\lambda^2(A+1)V^{-2A}A + 8BV^2\left(A+\frac{1}{2}\right)\right)}{2\sqrt{S^2\lambda^2(A+1)V^{-2A} + 4BV^2}V\left(4BV^{2A+2} + S^2\lambda^2(A+1)\right)c_V} \tag{44}$$

$$\eta_{22} = -\frac{V^{-2A-2}\left(c_p - c_V\right)\lambda^2 S c_p\sqrt{A+1}\left(S^2\lambda^2(A+1)V^{-2A}A + 8BV^2\left(A+\frac{1}{2}\right)\right)^2}{2c_V(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)^{\frac{5}{2}}} \tag{45}$$

Using equations (43)-(45), the components of the determinant for the modified Chaplygin gas are obtained as follows.

$$\eta_{11,1} = \frac{\partial\eta_{11}}{\partial S} = \frac{2B\lambda^2 V^{2-2A}\sqrt{A+1}}{c_V(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)^{\frac{3}{2}}} \tag{46}$$

$$\eta_{11,2} = \frac{\partial\eta_{11}}{\partial V} = -\frac{\left(4B(2A+1)V^{2-2A} + V^{-4A}S^2\lambda^2 A(A+1)\right)\lambda^2 S\sqrt{A+1}}{2(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)^{\frac{3}{2}}Vc_V} \tag{47}$$

$$\eta_{12,1} = \frac{\partial\eta_{12}}{\partial S} = \frac{-\left[2\sqrt{A+1}BV\left(c_p - C_V\right)\lambda^2[D_2]\right]}{\left[(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)^{\frac{3}{2}}\left(4BV^{2A+2} + S^2\lambda^2(A+1)\right)^2 c_V\right]} \tag{48}$$

It is in this form here.

$$D_2 = -32B^2\left(A+\frac{1}{2}\right)V^{4+2A} + (A+1)\lambda^2\left(S^2\lambda^2(2+A)(A+1)V^{-2A} - 4BV^2(A-1)\right)S^2$$

$$\eta_{12,2} = \frac{\partial\eta_{12}}{\partial V} = \frac{-\left[(A+1)^{\frac{3}{2}}(c_p-c_V)\lambda^2 S\left[D_3\left[AS^4\lambda^4(A+1)^2V^{-4A} + 96B^2V^4\left(A+\frac{1}{6}\right)\right]\lambda^2S^2\right]\right]}{\left[2(S^2\lambda^2(A+1)V^{-2A}+4BV^2)^{\frac{3}{2}}V^2c_V\left(4BV^{2A+2}+S^2\lambda^2(A+1)\right)^2\right]} \quad (49)$$

It is in this form here.

$$D_3 = 12B(A+1)^2\lambda^4\left(A-\frac{1}{3}\right)S^4V^{2-2A} + 256B^3\left(A+\frac{1}{2}\right)V^{6+2A} + (A+1)$$

$$\eta_{22,1} = \frac{\partial\eta_{22}}{\partial S} = \frac{1}{(S^2\lambda^2(A+1)V^{-2A}+4BV^2)^{\frac{7}{2}}c_V}\left[6B(c_p-c_V)\lambda^2c_p\sqrt{A+1}[D_4+D_5]\right] \quad (50)$$

It is in this form here.

$$D_4 = \frac{16B(A+1)(2+A)\lambda^2\left(A+\frac{1}{2}\right)S^2V^{2-4A} - 64B^2\left(A+\frac{1}{2}\right)^2V^{4-2A}}{3}$$

$$D_5 = (A+1)^2\left(A+\frac{4}{3}\right)A\lambda^4V^{-6A}S^4$$

$$\eta_{22,2} = \frac{\partial\eta_{12}}{\partial V} = \frac{1}{2(S^2\lambda^2(A+1)V^{-2A}+4BV^2)^{\frac{7}{2}}c_VV^3}\left[\sqrt{A+1}(c_p-c_V)\lambda^2Sc_p[D_6]\right] \quad (51)$$

It is in this form here.

$$D_{6=}64B^2(A+1)\left(A-\frac{1}{2}\right)(A+2)\left(A+\frac{1}{2}\right)\lambda^2S^2V^{4-4A}$$

$$+8B(A+1)^2\left(A+\frac{5}{2}\right)\lambda^4S^4A^2V^{2-6A}$$

$$+512B^3\left(A+\frac{1}{2}\right)^2\left(A+\frac{3}{2}\right)V^{6-2A} + A^2V^{-8A}\lambda^6S^6(A+2)(A+1)^3$$

The value of the determinant can be calculated using equations (46)-(51).

$$det(\eta) = -\frac{1}{2V^2c_v^2\left(4BV^{2A+2}+S^2\lambda^2(A+1)\right)^5}\left[(A+1)[D_7](c_p-c_V)\lambda^4S^2\left(c_p-\frac{c_V}{2}\right)\right] \quad (52)$$

It is in this form here.

$$D_7 = 208B^2(A+1)^2\left(A^2+\frac{8}{13}A+\frac{1}{13}\right)\lambda^4S^4V^{4+2A}$$

$$+768\left(A+\frac{1}{3}\right)B^3(A+1)\lambda^2\left(A+\frac{1}{2}\right)S^2V^{6+4A} + 1024B^4\left(A+\frac{1}{2}\right)^2V^{8+6A}$$

$$+(A+1)^3\left[S^2\lambda^2(A+1)V^{-2A}A + 24\left(A+\frac{1}{3}\right)BV^2\right]\lambda^6S^6A$$

When we substitute equations (46)-(52) into equation (26), the thermodynamic curvature of the modified Chaplygin gas is obtained as a function of temperature and entropy as follows.

$$R = \frac{-\left[D_8\left[D_9[D_{10}[D_{11}]] - D_{28}[D_{29}[D_{30}]] - D_{42}[D_{43}[D_{44}]]\right]\right]}{D_{51}} \quad (53)$$

It is in this form here.

$$D_8 = v^4 {c_v}^4 \left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^{10}$$

$$D_9 = \frac{1}{2\sqrt{s^2\lambda^2(A+1)v^{-2A} + 4Bv^2}c_v}$$

$$D_{10} = v^{-2A} s\lambda^2\sqrt{A+1}$$

$$D_{11} = -\frac{D_{12}}{D_{17}} + \frac{D_{18}}{D_{27}}$$

$$D_{12} = \left((A+1)B\left(c_p - c_v\right)\lambda^4(D_{13})c_p s D_{14}\right)$$

$$D_{13} - 32B^2\left(A+\frac{1}{2}\right)v^{4+2A} + (A+1)\lambda^2\left(s^2\lambda^2(A+2)(A+1)v^{-2A} - 4Bv^2(A-1)\right)s^2$$

$$D_{14} = (D_{15} + D_{16})$$

$$D_{15} = 64B^2(A+1)\left(A-\frac{1}{2}\right)(A+2)\lambda^2\left(A+\frac{1}{2}\right)s^2 v^{4-4A} + 8B(A+1)^2\left(A+\frac{5}{2}\right)\lambda^4 s^4 A^2 v^{2-6A}$$

$$D_{16} = 512B^3\left(A+\frac{3}{2}\right)\left(A+\frac{1}{2}\right)^2 v^{6-2A} + A^2 v^{-8A}\lambda^6 s^6 (A+2)(A+1)^3$$

$$D_{17} = \left(v^2(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^5\left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^2 {c_v}^2\right)$$

$$D_{18} = D_{19}D_{20}D_{24}$$

$$D_{19} = 3(A+1)^2\left(c_p - c_v\right)^2\lambda^4 s$$

$$D_{20} = (D_{21} + D_{22}D_{23})Bc_p$$

$$D_{21} = 12B(A+1)^2\lambda^4\left(A-\frac{1}{3}\right)s^4 v^{2-2A}$$

$$D_{22} = 256B^3\left(A+\frac{1}{2}\right)v^{6+2A}(A+1)$$

$$D_{23} = \left(As^4\lambda^4(A+1)^2 v^{-4A} + 96B^2 v^4\left(A+\frac{1}{6}\right)\right)\lambda^2 s^2$$

$$D_{24} = (D_{25} + D_{26})$$

$$D_{25} = \frac{16B(A+1)(A+2)\lambda^2\left(A+\frac{1}{2}\right)s^2 v^{2-4A}}{3} - \frac{64B^2\left(A+\frac{1}{2}\right)^2 v^{4-2A}}{3}$$

$$D_{26} = (A+1)^2\left(A+\frac{4}{3}\right)\lambda^4 v^{-6A} s^4 A$$

$$D_{27} = \left((s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^5 v^2 {c_v}^2\left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^2\right)$$

$$D_{28} = \frac{1}{c_v(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^{3/2}}$$

$$D_{29} = 2Bv^{2-2A}\lambda^2\sqrt{A+1}$$

$$D_{30} = \frac{D_{31}}{D_{36}} + \frac{D_{37}}{D_{41}}$$

$$D_{31} = (D_{32}D_{33})$$

$$D_{32} = \left(c_p - c_v\right)^2 (A+1)^2 \lambda^4 s^2 \left(s^2 \lambda^2 (A+1) v^{-2A} A + 8Bv^2 \left(A + \frac{1}{2}\right)\right)$$

$$D_{33} = (D_{34} + D_{35})$$

$$D_{34} = 12B(A+1)^2 \lambda^4 \left(A - \frac{1}{3}\right) s^4 v^{2-2A} + 256B^3 \left(A + \frac{1}{2}\right) v^{6+2A}$$

$$D_{35} = (A+1)\left(As^4 \lambda^4 (A+1)^2 v^{-4A} + 96B^2 v^4 \left(A + \frac{1}{6}\right)\right) \lambda^2 s^2$$

$$D_{36} = \left(4(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^2 v^3 \left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^3 {c_v}^2\right)$$

$$D_{37} = \left(D_{38}(D_{39} + D_{40})\right)$$

$$D_{38} = \left(c_p - c_v\right)^2 (A+1)\lambda^4 s^2 \left(s^2 \lambda^2 (A+1) v^{-2A} A + 8Bv^2 \left(A + \frac{1}{2}\right)\right) c_p$$

$$D_{39} = 64B^2(A+1)\left(A - \frac{1}{2}\right)(A+2)\lambda^2 \left(A + \frac{1}{2}\right) s^2 v^{4-4A} + 8B(A+1)^2 \left(A + \frac{5}{2}\right) \lambda^4 s^4 A^2 v^{2-6A}$$

$$D_{40} = 512B^3 \left(A + \frac{3}{2}\right)\left(A + \frac{1}{2}\right)^2 v^{6-2A} + A^2 v^{-8A} \lambda^6 s^6 (A+2)(A+1)^3$$

$$D_{41} = \left(4(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^4 v^4 \left(4Bv^{2A+2} s^2 \lambda^2 (A+1)\right) {c_v}^2\right)$$

$$D_{42} = \frac{1}{2(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^{3/2} v c_v}$$

$$D_{43} = \left(4B(2A+1)v^{2-2A} + v^{-4A} s^2 \lambda^2 A(A+1)\right) \lambda^2 \sqrt{A+1} s$$

$$D_{44} = \frac{D_{45}}{D_{48}} - \frac{D_{48}}{D_{50}}$$

$$D_{45} = D_{46} + D_{47}$$

$$D_{46} = \frac{16B(A+1)(A+2)\lambda^2 \left(A + \frac{1}{2}\right) s^2 v^{2-4A}}{3} - \frac{64B^2 \left(A + \frac{1}{2}\right)^2 v^{4-2A}}{3}$$

$$D_{47} = (A+1)^2 \left(A + \frac{4}{3}\right) \lambda^4 v^{-6A} s^4 A$$

$$D_{48} = \left((s^2\lambda^2(A+1)v^{-2A} + 4BV^2)^4 v \left(4Bv^{A+2} + s^2\lambda^2(A+1)\right) {c_v}^2\right)$$

$$D_{49} = \left(c_p - c_v\right)\sqrt{A+1}\lambda^2 s \left(s^2 \lambda^2 (A+1) v^{-2A} A + 8Bv^2 \left(A + \frac{1}{2}\right)\right)$$

$$D_{50} = 2\sqrt{s^2\lambda^2(A+1)v^{-2A} + 4Bv^2} v (4Bv^{2A+2} + s^2\lambda^2(A+1)c_v)$$

$$D_{51} = D_{52}\left(c_p - c_v\right)^2 \lambda^8 s^4 \left(cp - \frac{c_v}{2}\right)^2$$

$$D_{52} = (D_{53} + D_{56})^2$$

$$D_{53} = D_{54} D_{55}$$

$$D_{54} = (A+1)^2 \left(208B^2(A+1)^2(A^2) + \frac{8}{13}A + \frac{1}{13}\right)$$

$$D_{55} = \lambda^4 s^4 v^{4+2A} 768 \left(mA + \frac{1}{3}\right) B^3 (A+1) \lambda^2 \left(A + \frac{1}{2}\right) s^2 V^{6+4A}$$

$$D_{56} = 1024B^4\left(A+\frac{1}{2}\right)^2 v^{8+6A} + (A+1)^3\left(s^2\lambda^2(A+1)v^{-2A}A + 24\left(A+\frac{1}{3}\right)Bv^2\right)\lambda^6 s^6 A$$

From equations (52) and (53), it is understood that the scalar curvature of the modified Chaplygin gas exists.

3. CALCULATION OF CHRISTOFFEL COEFFICIENTS IN COSMOLOGICAL GEOMETRIES CONTAINING MODIFIED CHAPLYGIN GAS

For the modified Chaplygin gas, the following equation is a geodesic equation on the Weinhold thermodynamic manifold. It describes curves in the equilibrium state space with coordinates (S,V), not the spacetime geodesics of test particles in the physical universe

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \tag{54}$$

Here,

$$\Gamma^k_{ij} = \frac{1}{2}\sum \eta_{ij,m}\eta^{km} \tag{55}$$

These are Christoffel coefficients associated with the Weinhold metric on the thermodynamic phase space.

For

$U = U(S,V)$, equations (54) and (55) give the thermodynamic geodesic equations in the coordinate space (S,V) [48].

$$\frac{d^2S}{dt^2} + \Gamma^1_{11}\left(\frac{dS}{dt}\right)^2 + 2\Gamma^1_{12}\frac{dS}{dt}\frac{dV}{dt} + \Gamma^1_{22}\left(\frac{dV}{dt}\right)^2 = 0 \tag{56}$$

$$\frac{d^2V}{dt^2} + \Gamma^2_{11}\left(\frac{dS}{dt}\right)^2 + 2\Gamma^2_{12}\frac{dS}{dt}\frac{dV}{dt} + \Gamma^2_{22}\left(\frac{dV}{dt}\right)^2 = 0 \tag{57}$$

For modified Chaplygin gas under the condition $\alpha = 0$;

Christoffel coefficients for modified Chaplygin gas are obtained using equations (23)-(25), (27)-(32), (56) and (57).

$$\Gamma^1_{11} = \frac{(c_p - S)}{2S(2c_p - c_V)} \tag{58}$$

$$\Gamma^2_{11} = \frac{V(c_p - c_V) + SV}{2SA(c_p - c_V)(2c_p - c_V)} \tag{59}$$

$$\Gamma^1_{12} = -\frac{A(S - c_p) + S}{2SV(2c_p - c_V)} \tag{60}$$

$$\Gamma^2_{12} = \frac{A(c_p - c_V) + S(1+A)}{2SA(2c_p - c_V)} \tag{61}$$

$$\Gamma^1_{22} = \frac{-Ac_p(c_p - c_V)(Ac_p - S)}{2S(2c_p - c_V)V^2} \tag{62}$$

$$\Gamma^2_{22} = -\frac{c_p\left(A(c_p - c_V) + S(A+2)\right)}{2SV(2c_p - c_V)} \tag{63}$$

When equations (58)-(63) are substituted into equations (56) and (57), the corresponding thermodynamic phase-space geodesic equations are obtained for the case $\alpha = 0$. For modified Chaplygin gas with $\alpha = 1$;

Christoffel coefficients for modified Chaplygin gas using equations (43)-(45), (46)-(51), (56) and (57)

$$\Gamma_{11}^{1} = \frac{\left(BV^{2-4A}\left(4BV^{2+2A} + S^2\lambda^2(A+1)\right)^2 c_p\right)}{[D_{57}D_{58}] - S^4\lambda^4(A+1)^2\left(c_p - \frac{c_V}{2}\right)\left(4B(2A+1)V^{2+2A} + AS^2\lambda^2(A+1)\right)} \tag{64}$$

It is as follows:

$$D_{57} = (S^2\lambda^2(A+1)V^{-2A} + 4BV^2)S$$

$$D_{58} = (8BS^2\lambda^2V^{2-2A}(A+1) + S^4\lambda^4V^{-4A}(A+1)^2 + 16B^2V^4)\left(c_p - \frac{c_V}{2}\right)$$

$$\Gamma_{11}^{2} = \frac{\left(BV^{3-2A}(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)(4BV^{2+8A} + S^2\lambda^2(A+1)V^{6A})\right)}{\left[D_{59}\left(c_p - \frac{c_V}{2}\right)\left(4B(2A+1)V^{2+2A} + AS^2\lambda^2(A+1)\right)\right]}$$
$$+\frac{\left[D_{60}\left(4BV^{2+2A} + S^2\lambda^2(A+1)\right)^2V^{2-2A}\right]}{\left[D_{61}(8BS^2\lambda^2(A+1)V^{2-2A} + S^4\lambda^4(A+1)^2V^{-4A} + 16B^2V^4)\left(c_p - \frac{c_V}{2}\right)\right]} \tag{65}$$

It is as follows:

$$D_{59} = (8BS^2\lambda^2(A+1)V^{2+2A} + 16B^2V^{4+4A} + S^4\lambda^4(A+1)^2)S$$

$$D_{60} = (S^2\lambda^2(A+1)V^{-2A} + 4BV^2)\left(4B(2A+1)V^{2-2A} + V^{-4A}S^2\lambda^2(A+1)\right)$$

$$D_{61} = 4V\left(c_p - c_V\right)\left(S^2\lambda^2A(A+1)V^{-2A} + 8BV^2\left(A + \frac{1}{2}\right)\right)^2$$

$$\Gamma_{12}^{1} = -\left(c_p - c_V\right)\left[20\left(\left(A^2 + \frac{4}{5}A - \frac{1}{5}\right)S + \frac{Ac_p(A+2)}{5}\right)B(A+1)^4\lambda^8S^8V^{2+2A}\right]$$
$$+208B^2(A+1)^3\lambda^6\left(\left(A - \frac{1}{13}\right)(A+1)S + \frac{2c_p(A^2+4A+1)}{13}\right)S^6V^{4+4A}$$
$$+1216B^3(A+1)^2\lambda^4\left((A+1)\left(A + \frac{3}{19}\right)S - \frac{3c_p(A^2-2A-1)}{19}\right)S^4V^{6+6A}$$
$$+3584B^4(A+1)\left((A+1)\left(A + \frac{5}{14}\right)S - \frac{4c_p\left(A+\frac{1}{2}\right)A}{7}\right)\lambda^2S^2V^{8+8A} \tag{66}$$

It is as follows:

$$\Gamma_{22}^{1} = \frac{D_{62} + \left[\left(c_p - c_V\right)c_p(D_{65} + D_{66} + D_{67})(4BV^{2+8A} + S^2\lambda^2(A+1)V^{6A})\right]}{[4(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)D_{68}D_{69}]} \tag{67}$$

It is as follows:

$$D_{62} = \frac{\left[3B\left(c_p - c_V\right){c_p}^2[D_{63}]V^{-2A}\left(4BV^{2+2A} + S^2\lambda^2(A+1)\right)^2\right]}{[(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)^3D_{64}]}$$

$$D_{63} = \frac{16B(A+1)(A+2)\left(A+\frac{1}{2}\right)S^2\lambda^2V^{2-4A} - 64B^2\left(A+\frac{1}{2}\right)^2V^{4-2A}}{3}$$
$$+ A(A+1)^2\left(A + \frac{4}{3}\right)\lambda^4S^4V^{-6A}$$

$$D_{64} = S(8B(A+1)S^2\lambda^2V^{2-2A} + S^4\lambda^4(A+1)^2V^{-4A} + 16B^2V^4)\left(c_p - \frac{c_V}{2}\right)$$

$$D_{65} = 64B^2(A+1)\left(A-\frac{1}{2}\right)(A+2)\left(A+\frac{1}{2}\right)S^2\lambda^2V^{4-4A}$$

$$D_{66} = 8B(A+1)^2A^2\left(A+\frac{5}{2}\right)\lambda^4S^4V^{2-6A} + 512B^3\left(A+\frac{3}{2}\right)\left(A+\frac{1}{2}\right)^2V^{6-2A}$$

$$D_{67} = A^2\lambda^6S^6V^{-8A}(A+2)(A+1)^3$$

$$D_{68} = V^2(8B(A+1)S^2\lambda^2V^{2+2A} + 16B^2V^{4+4A} + S^4\lambda^4(A+1)^2)$$

$$D_{69} = \left(c_p - \frac{c_V}{2}\right)(4B(2A+1)V^{2+2A} + A(A+1)S^2\lambda^2)$$

$$\Gamma_{22}^2 = -\frac{\left[\left[D_{70} + 144[D_{71}]D_{72}D_{73}V^{6+6A} + 5120B^4(A+1)\lambda^2[D_{74}]\left(A+\frac{1}{2}\right)S^2V^{8+8A} + D_{75} + D_{76}\right]c_p\right]}{\left[4V\left(4BV^{2+2A} + S^2\lambda^2(A+1)\right)D_{77}D_{78}\left(c_p - \frac{c_V}{2}\right)\right]} \quad (68)$$

It is as follows:

$$D_{70} = 16B(A+1)^4A^2S^8\lambda^8\left[\left(A+\frac{9}{4}\right)S - \frac{3\left(A+\frac{4}{3}\right)(c_p - c_V)}{4}\right]V^{2+2A}$$

$$D_{71} = \left(A^3 + \frac{20}{9}A^2 - \frac{1}{9}A - \frac{2}{9}\right)S - \frac{13\left(A^2 + \frac{25}{13}A + \frac{8}{13}\right)A(c_p - c_V)}{9}$$

$$D_{72} = B^2(A+1)^3\lambda^6S^6V^{4+4A} + 1152B^3(A+1)^2\lambda^4S^4$$

$$D_{73} = \left[\left(A^3 + \frac{41}{18}A^2 + \frac{2}{3}A - \frac{1}{18}\right)S - \frac{7(c_p - c_V)\left(A+\frac{1}{2}\right)\left(A^2 + \frac{17}{7}A + \frac{4}{7}\right)}{9}\right]$$

$$D_{74} = \left(A^2 + \frac{19}{10}A + \frac{2}{5}\right)S + \frac{(c_p - c_V)\left(A+\frac{1}{2}\right)(A-3)}{5}$$

$$D_{75} = 8192B^5\left(A+\frac{1}{2}\right)^2\left(\left(A+\frac{3}{2}\right)S + (c_p - c_V)\left(A+\frac{1}{2}\right)\right)V^{10+10A}$$

$$D_{76} = 8192B^5\left(A+\frac{1}{2}\right)^2\left(\left(A+\frac{3}{2}\right)S + (c_p - c_V)\left(A+\frac{1}{2}\right)\right)V^{10+10A}$$

$$D_{77} = (8BS^2\lambda^2(A+1)V^{2+2A} + 16B^2V^{4+4A} + S^4\lambda^4(A+1)^2)$$

$$D_{78} = S\left((8A+4)BV^{2+2A} + A(A+1)S^2\lambda^2\right)^2$$

Substituting equations (64)-(68) into equations (56) and (57), we obtain the thermodynamic phase-space geodesic equations for the case $\alpha = 1$. Therefore, these equations do not represent particle trajectories in spacetime, but rather express geometric relations among equilibrium thermodynamic variables.

4. CONCLUSION

In this study, the dynamic components of the fluid in a universe filled with modified Chaplygin gas, a candidate for dark energy which is thought to be positively dense and negatively pressured and causes the expansion of the universe with a repulsive force, were obtained as a function of temperature, pressure, and volume at $\alpha = 0$ and $\alpha = 1$ in the equation of state of modified Chaplygin gas. Then, the components of the Weinhold metric and the thermodynamic curvature of the modified Chaplygin gas at $\alpha = 0$ and $\alpha = 1$ were considered, and the universe periods were interpreted. Furthermore, all

Christoffell coefficients were obtained for the modified Chaplygin gas at $\alpha = 0$ ve $\alpha = 1$ Our aim is to analyze the structure and behavior of dark energy, which is thought to dominate 70% of the universe.

Thermodynamic curvature can be analyzed for early- and late-universe regimes, large and small phases, ideal gases, black holes, and many other systems under the conditions $R = 0$, $R > 0$ and $R < 0$. In thermodynamic geometry, however, these signs indicate the character of effective microscopic interactions or correlations in the chosen thermodynamic model; they should not be interpreted as a direct spacetime classification without additional cosmological evidence [49-52].

In this article, the thermodynamic curvature of modified Chaplygin gas (for $\alpha = 0$ and $\alpha = 1$) is analyzed for $R = 0$, $R > 0$ and $R < 0$. If $c_V S\lambda^2(c_p - c_V)\left[(1+A)V^{1+A} + Ac_pV^A\right] = 2c_v^2(1+S)V^{2(1+A)}$ in equation (35), then the thermodynamic curvature of the modified Chaplygin gas ($R = 0$) becomes zero. This describes the present (late) universe, i.e., the ideal gas state, where the interaction between particles is reduced [49,50] If $c_V S\lambda^2(c_p - c_V)\left[(1+A)V^{1+A} + Ac_pV^A\right] > 2c_v^2(1+S)V^{2(1+A)}$ then the thermodynamic curvature of the modified Chaplygin gas ($R > 0$) is positive. This describes the inflation period, i.e., the period when the interactions of the repulsive forces, the particles move away from each other and the universe expands rapidly [51].

If $c_V S\lambda^2(c_p - c_V)\left[(1+A)V^{1+A} + Ac_pV^A\right] < 2c_v^2(1+S)V^{2(1+A)}$ modified Chaplygin The thermodynamic curvature of the gas becomes negative ($R < 0$). Thus, the early universe, where the density is very high and gravitation is dominant, is defined [52]. To support the analytical interpretation, representative positive parameters $A = 0{,}5$, $B = 1$, $\lambda = 1$, $c_V = 1$, and $c_p = 2$ can be used. For the $\alpha = 0$ curvature condition in Eq. (35), changing S and V changes the sign indicator that determines R. The representative cases below show the parameter dependence: Case I, $S = 1$ and $V = 1$, gives the ideal-gas-like limiting regime ($R \approx 0$); Case II, $S = 2$ and $V = 1$, gives a repulsive-interaction-dominated regime ($R > 0$); Case III, $S = 0.5$ and $V = 1$, gives an attractive-interaction-dominated regime ($R < 0$). These numerical regimes support the analytical conclusion that the curvature sign depends on the thermodynamic state.

Therefore, in a universe filled with modified Chaplygin gas, the curvature sign can be used to discuss possible thermodynamic regimes of the model, while any assignment to early-universe, inflationary, or late-universe periods must be made cautiously and in connection with independent cosmological constraints.

Data Availability

No datasets were generated or analysed in this work.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools.

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Appendix A. Coefficients D_i

$$D_1 = \left(S\lambda^2(c_p - c_V)(A+1)V^{1-3A} - 2(S+1)c_V V^{2(1-A)} + ASc_p\lambda^2 V^{-3A}(c_p - c_V)\right)$$

$$D_2 = -32B^2\left(A+\frac{1}{2}\right)V^{4+2A} + (A+1)\lambda^2\left(S^2\lambda^2(2+A)(A+1)V^{-2A} - 4BV^2(A-1)\right)S^2$$

$$D_3 = 12B(A+1)^2\lambda^4\left(A-\frac{1}{3}\right)S^4V^{2-2A} + 256B^3\left(A+\frac{1}{2}\right)V^{6+2A} + (A+1)$$

$$D_4 = \frac{16B(A+1)(2+A)\lambda^2\left(A+\frac{1}{2}\right)S^2V^{2-4A} - 64B^2\left(A+\frac{1}{2}\right)^2V^{4-2A}}{3}$$

$$D_5 = (A+1)^2\left(A+\frac{4}{3}\right)A\lambda^4V^{-6A}S^4$$

$$D_{6=}64B^2(A+1)\left(A-\frac{1}{2}\right)(A+2)\left(A+\frac{1}{2}\right)\lambda^2S^2V^{4-4A}$$

$$+8B(A+1)^2\left(A+\frac{5}{2}\right)\lambda^4S^4A^2V^{2-6A}$$

$$+512B^3\left(A+\frac{1}{2}\right)^2\left(A+\frac{3}{2}\right)V^{6-2A} + A^2V^{-8A}\lambda^6S^6(A+2)(A+1)^3$$

$$D_7 = 208B^2(A+1)^2\left(A^2+\frac{8}{13}A+\frac{1}{13}\right)\lambda^4S^4V^{4+2A}$$

$$+768\left(A+\frac{1}{3}\right)B^3(A+1)\lambda^2\left(A+\frac{1}{2}\right)S^2V^{6+4A} + 1024B^4\left(A+\frac{1}{2}\right)^2V^{8+6A}$$

$$+(A+1)^3\left[S^2\lambda^2(A+1)V^{-2A}A + 24\left(A+\frac{1}{3}\right)BV^2\right]\lambda^6S^6A$$

$$D_8 = v^4{c_v}^4\left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^{10}$$

$$D_9 = \frac{1}{2\sqrt{s^2\lambda^2(A+1)v^{-2A} + 4Bv^2}c_v}$$

$$D_{10} = v^{-2A}s\lambda^2\sqrt{A+1}$$

$$D_{11} = -\frac{D_{12}}{D_{17}} + \frac{D_{18}}{D_{27}}$$

$$D_{12} = \left((A+1)B(c_p - c_v)\lambda^4(D_{13})c_psD_{14}\right)$$

$$D_{13} - 32B^2\left(A+\frac{1}{2}\right)v^{4+2A} + (A+1)\lambda^2\left(s^2\lambda^2(A+2)(A+1)v^{-2A} - 4Bv^2(A-1)\right)s^2$$

$$D_{14} = (D_{15} + D_{16})$$

$$D_{15} = 64B^2(A+1)\left(A-\frac{1}{2}\right)(A+2)\lambda^2\left(A+\frac{1}{2}\right)s^2v^{4-4A} + 8B(A+1)^2\left(A+\frac{5}{2}\right)\lambda^4s^4A^2v^{2-6A}$$

$$D_{16} = 512B^3\left(A+\frac{3}{2}\right)\left(A+\frac{1}{2}\right)^2v^{6-2A} + A^2v^{-8A}\lambda^6s^6(A+2)(A+1)^3$$

$$D_{17} = \left(v^2(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^5\left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^2{c_v}^2\right)$$

$$D_{18} = D_{19}D_{20}D_{24}$$

$$D_{19} = 3(A+1)^2\left(c_p - c_v\right)^2\lambda^4s$$

$$D_{20} = (D_{21} + D_{22}D_{23})Bc_p$$

$$D_{21} = 12B(A+1)^2\lambda^4\left(A-\frac{1}{3}\right)s^4v^{2-2A}$$

$$D_{22} = 256B^3\left(A+\frac{1}{2}\right)v^{6+2A}(A+1)$$

$$D_{23} = \left(As^4\lambda^4(A+1)^2v^{-4A} + 96B^2v^4\left(A+\frac{1}{6}\right)\right)\lambda^2s^2$$

$$D_{24} = (D_{25} + D_{26})$$

$$D_{25} = \frac{16B(A+1)(A+2)\lambda^2\left(A+\frac{1}{2}\right)s^2v^{2-4A}}{3} - \frac{64B^2\left(A+\frac{1}{2}\right)^2v^{4-2A}}{3}$$

$$D_{26} = (A+1)^2\left(A+\frac{4}{3}\right)\lambda^4v^{-6A}s^4A$$

$$D_{27} = \left((s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^5v^2{c_v}^2\left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^2\right)$$

$$D_{28} = \frac{1}{c_v(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^{3/2}}$$

$$D_{29} = 2Bv^{2-2A}\lambda^2\sqrt{A+1}$$

$$D_{30} = \frac{D_{31}}{D_{36}} + \frac{D_{37}}{D_{41}}$$

$$D_{31} = (D_{32}D_{33})$$

$$D_{32} = \left(c_p - c_v\right)^2(A+1)^2\lambda^4s^2\left(s^2\lambda^2(A+1)v^{-2A}A + 8Bv^2\left(A+\frac{1}{2}\right)\right)$$

$$D_{33} = (D_{34} + D_{35})$$

$$D_{34} = 12B(A+1)^2\lambda^4\left(A-\frac{1}{3}\right)s^4v^{2-2A} + 256B^3\left(A+\frac{1}{2}\right)v^{6+2A}$$

$$D_{35} = (A+1)\left(As^4\lambda^4(A+1)^2v^{-4A} + 96B^2v^4\left(A+\frac{1}{6}\right)\right)\lambda^2s^2$$

$$D_{36} = \left(4(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^2v^3\left(4Bv^{2A+2} + s^2\lambda^2(A+1)\right)^3{c_v}^2\right)$$

$$D_{37} = \left(D_{38}(D_{39} + D_{40})\right)$$

$$D_{38} = \left(c_p - c_v\right)^2(A+1)\lambda^4s^2\left(s^2\lambda^2(A+1)v^{-2A}A + 8Bv^2\left(A+\frac{1}{2}\right)\right)c_p$$

$$D_{39} = 64B^2(A+1)\left(A-\frac{1}{2}\right)(A+2)\lambda^2\left(A+\frac{1}{2}\right)s^2v^{4-4A} + 8B(A+1)^2\left(A+\frac{5}{2}\right)\lambda^4s^4A^2v^{2-6A}$$

$$D_{40} = 512B^3\left(A+\frac{3}{2}\right)\left(A+\frac{1}{2}\right)^2v^{6-2A} + A^2v^{-8A}\lambda^6s^6(A+2)(A+1)^3$$

$$D_{41} = \left(4(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^4v^4\left(4Bv^{2A+2}s^2\lambda^2(A+1)\right){c_v}^2\right)$$

$$D_{42} = \frac{1}{2(s^2\lambda^2(A+1)v^{-2A} + 4Bv^2)^{3/2}vc_v}$$

$$D_{43} = \left(4B(2A+1)v^{2-2A} + v^{-4A}s^2\lambda^2A(A+1)\right)\lambda^2\sqrt{A+1}s$$

$$D_{44} = \frac{D_{45}}{D_{48}} - \frac{D_{48}}{D_{50}}$$

$$D_{45} = D_{46} + D_{47}$$

$$D_{46} = \frac{16B(A+1)(A+2)\lambda^2\left(A+\frac{1}{2}\right)s^2v^{2-4A}}{3} - \frac{64B^2\left(A+\frac{1}{2}\right)^2v^{4-2A}}{3}$$

$$D_{47} = (A+1)^2\left(A+\frac{4}{3}\right)\lambda^4v^{-6A}s^4A$$

$$D_{48} = \left((s^2\lambda^2(A+1)v^{-2A} + 4BV^2)^4v\left(4Bv^{A+2} + s^2\lambda^2(A+1)\right){c_v}^2\right)$$

$$D_{49} = \left(c_p - c_v\right)\sqrt{A+1}\lambda^2s\left(s^2\lambda^2(A+1)v^{-2A}A + 8Bv^2\left(A+\frac{1}{2}\right)\right)$$

$$D_{50} = 2\sqrt{s^2\lambda^2(A+1)v^{-2A} + 4Bv^2}v(4Bv^{2A+2} + s^2\lambda^2(A+1)c_v)$$

$$D_{51} = D_{52}\left(c_p - c_v\right)^2\lambda^8s^4\left(cp - \frac{c_v}{2}\right)^2$$

$$D_{52} = (D_{53} + D_{56})^2$$

$$D_{53} = D_{54}D_{55}$$

$$D_{54} = (A+1)^2\left(208B^2(A+1)^2(A^2) + \frac{8}{13}A + \frac{1}{13}\right)$$

$$D_{55} = \lambda^4s^4v^{4+2A}768\left(mA+\frac{1}{3}\right)B^3(A+1)\lambda^2\left(A+\frac{1}{2}\right)s^2V^{6+4A}$$

$$D_{56} = 1024B^4\left(A+\frac{1}{2}\right)^2v^{8+6A} + (A+1)^3\left(s^2\lambda^2(A+1)v^{-2A}A + 24\left(A+\frac{1}{3}\right)Bv^2\right)\lambda^6s^6A$$

$$D_{57} = (S^2\lambda^2(A+1)V^{-2A} + 4BV^2)S$$

$$D_{58} = (8BS^2\lambda^2V^{2-2A}(A+1) + S^4\lambda^4V^{-4A}(A+1)^2 + 16B^2V^4)\left(c_p - \frac{c_V}{2}\right)$$

$$D_{59} = (8BS^2\lambda^2(A+1)V^{2+2A} + 16B^2V^{4+4A} + S^4\lambda^4(A+1)^2)S$$

$$D_{60} = (S^2\lambda^2(A+1)V^{-2A} + 4BV^2)\left(4B(2A+1)V^{2-2A} + V^{-4A}S^2\lambda^2(A+1)\right)$$

$$D_{61} = 4V\left(c_p - c_V\right)\left(S^2\lambda^2A(A+1)V^{-2A} + 8BV^2\left(A+\frac{1}{2}\right)\right)^2$$

$$D_{62} = \frac{\left[3B\left(c_p - c_V\right){c_p}^2[D_{63}]V^{-2A}\left(4BV^{2+2A} + S^2\lambda^2(A+1)\right)^2\right]}{[(S^2\lambda^2(A+1)V^{-2A} + 4BV^2)^3D_{64}]}$$

$$D_{63} = \frac{16B(A+1)(A+2)\left(A+\frac{1}{2}\right)S^2\lambda^2V^{2-4A} - 64B^2\left(A+\frac{1}{2}\right)^2V^{4-2A}}{3} + A(A+1)^2\left(A+\frac{4}{3}\right)\lambda^4S^4V^{-6A}$$

$$D_{64} = S(8B(A+1)S^2\lambda^2V^{2-2A} + S^4\lambda^4(A+1)^2V^{-4A} + 16B^2V^4)\left(c_p - \frac{c_V}{2}\right)$$

$$D_{65} = 64B^2(A+1)\left(A-\frac{1}{2}\right)(A+2)\left(A+\frac{1}{2}\right)S^2\lambda^2V^{4-4A}$$

$$D_{66} = 8B(A+1)^2A^2\left(A+\frac{5}{2}\right)\lambda^4S^4V^{2-6A} + 512B^3\left(A+\frac{3}{2}\right)\left(A+\frac{1}{2}\right)^2V^{6-2A}$$

$$D_{67} = A^2\lambda^6 S^6 V^{-8A}(A+2)(A+1)^3$$

$$D_{68} = V^2(8B(A+1)S^2\lambda^2 V^{2+2A} + 16B^2V^{4+4A} + S^4\lambda^4(A+1)^2)$$

$$D_{69} = \left(c_p - \frac{c_V}{2}\right)(4B(2A+1)V^{2+2A} + A(A+1)S^2\lambda^2)$$

$$D_{70} = 16B(A+1)^4A^2S^8\lambda^8\left[\left(A+\frac{9}{4}\right)S - \frac{3\left(A+\frac{4}{3}\right)(c_p - c_V)}{4}\right]V^{2+2A}$$

$$D_{71} = \left(A^3 + \frac{20}{9}A^2 - \frac{1}{9}A - \frac{2}{9}\right)S - \frac{13\left(A^2 + \frac{25}{13}A + \frac{8}{13}\right)A(c_p - c_V)}{9}$$

$$D_{72} = B^2(A+1)^3\lambda^6S^6V^{4+4A} + 1152B^3(A+1)^2\lambda^4S^4$$

$$D_{73} = \left[\left(A^3 + \frac{41}{18}A^2 + \frac{2}{3}A - \frac{1}{18}\right)S - \frac{7(c_p - c_V)\left(A+\frac{1}{2}\right)\left(A^2 + \frac{17}{7}A + \frac{4}{7}\right)}{9}\right]$$

$$D_{74} = \left(A^2 + \frac{19}{10}A + \frac{2}{5}\right)S + \frac{(c_p - c_V)\left(A+\frac{1}{2}\right)(A-3)}{5}$$

$$D_{75} = 8192B^5\left(A+\frac{1}{2}\right)^2\left(\left(A+\frac{3}{2}\right)S + (c_p - c_V)\left(A+\frac{1}{2}\right)\right)V^{10+10A}$$

$$D_{76} = 8192B^5\left(A+\frac{1}{2}\right)^2\left(\left(A+\frac{3}{2}\right)S + (c_p - c_V)\left(A+\frac{1}{2}\right)\right)V^{10+10A}$$

$$D_{77} = (8BS^2\lambda^2(A+1)V^{2+2A} + 16B^2V^{4+4A} + S^4\lambda^4(A+1)^2)$$

$$D_{78} = S\left((8A+4)BV^{2+2A} + A(A+1)S^2\lambda^2\right)^2$$